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ON THE NUMBER OF SOLUTIONS TO POLYNOMIAL SYSTEMS OF EQUATIONS

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ON THE NUMBER OF SOLUTIONS TO POLYNOMIAL SYSTEMS OF EQUATIONS

C. B. Garcia* and T. Y. Li**

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ABSTRACT

It is shown that for almost every system of n polynomial equations in n complex variables, the number of solutions q is ~~equal to q~~ shown by an equation. $\equiv \prod_{i=1} q_i$, where q_i is the degree of equation i. The proof of this result is done in such a way that all q solutions can be explicitly calculated for almost all such systems.

It is further shown that if the polynomial system obtained by retaining only the terms of degree q_i in each equation i has only the trivial solution, then the number of solutions is equal to q.

AMS (MOS) Subject Classifications - 65H10, 69.32

Key Words - Homotopy, path-following, polynomial systems, systems of equations, complex variables, algebraic systems

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SIGNIFICANCE AND EXPLANATION

The existence and uniqueness of the solution of systems of linear algebraic equations $Ax = b$ depends on the ranks of A and the augmented matrix (A,b). The theory is particularly simple when A is a square matrix: a unique solution exists in "most" cases, namely when $Ax = 0$ has the trivial solution $x = 0$.

Although of obvious practical and theoretical importance, analogous questions for systems of n polynomial equations in n unknowns do not seem to have been settled definitively, except for the case $n = 1$, which is the classical theorem that every polynomial equation of degree q has exactly q complex roots. This paper shows (as one might expect) that for "almost all" systems of n polynomials in n complex variables, the number of solutions is equal to the product of the powers of the highest ordered terms in each of the n equations. A sufficient condition is given for this situation to occur. In this case the solutions can be calculated by a path-following procedure.

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ON THE NUMBER OF SOLUTIONS TO POLYNOMIAL SYSTEMS OF EQUATIONS

C. B. Garcia* and T. Y. Li**

§1. INTRODUCTION

The fundamental theorem of algebra states that every polynomial equation of one complex variable has q roots, counting multiplicities, where q is the degree of the polynomial. This paper generalizes this theorem to systems of n polynomial equations in n complex variables. It is shown that for "almost all" systems, the number of solutions is equal to $q \equiv \prod_{i=1}^{n} q_i$, where q_i is the degree (the power of the highest ordered term) of the ith equation. Emphasis is made on being able to calculate all solutions. Hence, the proof is written such that these q solutions can be calculated if desired by the path-following method first described in [7]. (Extensions of this approach may be found in [3, 5, 8].) Moreover, the method of proof can be used to furnish a new constructive and topological proof of the classical algebraic theorems of Bezout [11], and Noether and van der Waerden [9].

By "almost all", we mean that the property is "typical" or "generic", although there may be "exceptional" cases. However, this does not necessarily mean that the exceptional cases are unimportant. For example, for linear systems, our theorem would reduce to a statement that the solution is almost always unique. Yet, this does not mean that one can ignore the special cases when there is no solution or an infinite number of solutions. Hence, because the "exceptional" cases are important, we study in section 3 conditions where one can state with certainty the number of solutions to a particular problem. There, a sufficient condition is given for the number of solutions to be exactly q.

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§2. GENERICITY

Let $P : C^n \to C^n$ be a polynomial system, where C^n is the n-dimensional complex space. By a polynomial system, we mean that each term in every equation is of the form

$$a z_1^{r_1} z_2^{r_2}, \ldots, z_n^{r_n} \tag{2.1}$$

where a is a complex number, z_i a complex variable, and r_i a nonnegative integer. We are interested in the number of solutions $z = (z_1, z_2, \ldots, z_n)$ to

$$P(z) = 0 \ .$$

For each term (of the form (2.1)) in each equation i, consider the sum $r_1 + r_2 + \ldots + r_n$. Let q_i be the maximum sum in equation i. We assume $q_i > 0$, for all i. We call q_i the <u>degree</u> of P_i. We show that for almost all P, $q \equiv \prod_{i=1}^{n} q_i$ is the number of solutions to $P(z) = 0$.

By "almost all", we mean that the property is <u>generic</u>. In other words, imagine a class of problems

$$P(z,w) = 0 \tag{2.2}$$

where w is a parameter vector ranging over some complex space C^m. To each polynomial system is associated a parameter vector w, the correspondence to be made precise in a moment. It is best to think of w as a <u>random</u> vector governed by some probability measure on C^m. An "exceptional" set is then a set whose probability of occurrence is 0. A condition holds <u>generically</u> if it holds with probability 1. The theorem we show is

<u>Theorem 2.1</u>. For all w except in a set of measure zero in C^m, the system $P(z,w) = 0$ has exactly $q \equiv \prod_{i=1}^{n} q_i$ distinct solutions.

Let $P : C^n \to C^n$ be given. We define the parameter w to be the vector of <u>coefficients</u> that uniquely defines P. Given w, P is uniquely defined, and vice versa. The vector w belongs to some complex space C^m, for some m. The integer m is determined by the degrees q_i, all i.

We also distinguish certain coefficients of P. First, let $a = (a_{ij}) \in C^{n^2}$ be the entries of w such that

$$a_{ij} \text{ is the coefficient of } z_j^{q_i} \text{ in equation } i .$$

Then, let b be the entries of w such that $b_i \in C$ is the constant term of P_i, for all i. Finally, let c be those entries of P which are the coefficients of terms of degree q_i in equation i, for all i (excepting the a_{ij}'s) and d the rest of w. Hence, $w = (a,b,c,d)$ uniquely defines $P(z,w) = 0$.

The proof is shown by following the paths described by the homotopy $H : C^{n+m} \times R \to C^n$ where H is defined by

$$H_i(z,t,w) = (1-t)(z_i^{q_i} - 1) + tP_i(z,w) = 0 \qquad (2.3)$$
$$i = 1,2,\ldots,n .$$

For a fixed $w \in C^m$, starting from the $q \equiv \prod_{i=1}^{n} q_i$ trivial solutions when $t = 0$, we follow the solutions as t is gradually deformed to one (observe that the q trivial solutions need not be as chosen, but could be chosen arbitrarily). It will be shown that for almost all w, the solution set

$$H_w^{-1}(0) = \{(z,t) \mid H(z,t,w) = 0,\ 0 \le t \le 1\} \qquad (2.4)$$

consists of q distinct paths, where each path starts at a trivial solution at $t = 0$, and ends at a solution to $P(z,w) = 0$ at $t = 1$. Hence, for almost all w, $P(z,w) = 0$ has q distinct solutions.

Let $F : D \subset R^p \to R^s$ be a differentiable map. For any $x \in D$, $DF(x)$ will denote the Jacobian of F at x. If $x = (u,v)$ say, $D_uF(x)$ will denote the partial derivative of F with respect to u. If rank $DF(x) < s$, then x is a critical point for F. Otherwise, x is a regular point for F. If $y = F(x)$ for some critical point $x \in D$, y is said to be a critical value for F. Otherwise, y is a regular value of F.

In order that $H_w^{-1}(0)$ be consisting of paths, we shall require a regularity condition on H and on P. For then we can use the following theorem (shown in [7]) if a regularity condition holds.

Theorem 2.2. Given $w \in C^m$, let 0 be a regular value of $P(\cdot,w)$ and 0 a regular value of $H(\cdot,\cdot,w)$. Then $H_w^{-1}(0)$ consists of paths. Each solution in $H_w^{-1}(0)$ at at $t = 0$ or $t = 1$ is an endpoint of the path. Moreover, t is monotonically nonincreasing (nondecreasing) on each path.

The monotonicity of t in Theorem 2.2 implies that no path can be homeomorphic to a circle. Therefore, each solution of $H_w^{-1}(0)$ starting at $t = 0$ (or $t = 1$) will either go to infinity or go to a solution of $H_w^{-1}(0)$ at $t = 1$ ($t = 0$).

It will now be shown that for almost all w, no path of $H_w^{-1}(0)$ diverges to infinity on $\{(z,t) | 0 \le t \le 1\}$. Hence, for almost all w, $H_w^{-1}(0)$ consists of q paths. Each path connects a trivial solution at $t = 0$ to a solution of $P(z,w) = 0$ at $t = 1$. Hence $P(z,w) = 0$ has q solutions.

We recall the Transversality Theorem [1].

Theorem 2.3. For $D \subset R^p$, let $F : D \times R^k \to R^s$ be r-continuously differentiable, where $r > \max\{0, p - s\}$. Suppose $y \in R^s$ is a regular value of F. Then for all $u \in R^k$ except in a set of measure zero in R^k, y is a regular value of $F(\cdot,u)$.

Let us apply this theorem to $P(z,w) = 0$. Recall that w is partitioned into $w = (a,b,c,d)$.

Lemma 2.4. For all a,c, and d, and for all b except in a set of measure zero, 0 is a regular value of $P(\cdot,w)$ and 0 is a regular value of $H(\cdot,\cdot,w)$.

Proof: Clearly, 0 is a regular value of $P(\cdot,a,\cdot,c,d)$ since $D_b P = I$ is of full rank. Hence, by Theorem 2.3, 0 is a regular value of $P(\cdot,a,b,c,d)$ for almost all b. Similarly, 0 is a regular value of $H(\cdot,\cdot,a,\cdot,c,d)$. Hence for almost all b, 0 is a regular value of $H(\cdot,\cdot,a,b,c,d)$. ■

Next, given w let Q_i consist of the terms of $P_i(z,w)$ with degree q_i. Observe that the entries (a,c) of w uniquely identifies Q. In fact, we write Q as $Q(z,a,c) = 0$. Now define

$$G_i(z,t,a,c) = (1-t)z_i^{q_i} + tQ_i(z,a,c) = 0 \qquad (2.5)$$

$$i = 1,2,\ldots,n$$

Lemma 2.5. For arbitrary c and for all $a \in C^{n^2}$ except in a set of measure zero in C^{n^2}, 0 is a regular value of $G(\cdot,\cdot,a,c)$ on the domain set $\{(z,t) | z \ne 0\}$.

Proof: If $t = 0$, then $z = 0$ is the only solution for $G(\cdot,0,a,c) = 0$. So, let $t \ne 0$ and $z \ne 0$. Choose j such that $z_j \ne 0$. Then, differentiating G with respect to $a_{\cdot j} = (a_{1j},a_{2j},\ldots,a_{nj})$ we get $D_{a_{\cdot j}} G$ equal to a diagonal matrix with

diagonals $tz_j^{q_i}$, $i = 1,\ldots,n$. Hence DG has full rank and 0 is a regular value of $G(\cdot,\cdot,a,c)$ for almost all a. ∎

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1.

By Lemmas 2.4 and 2.5, for all $w \in C^m$ except in a set of measure zero in C^m, $P(z,w) = 0$ is such that 0 is a regular value of $P(\cdot,w)$ and $H(\cdot,\cdot,w)$ and 0 is a regular value of $G(\cdot,\cdot,a,c)$ on $\{(z,t)|z \neq 0\}$. (In fact only the entries a,b need be perturbed). Let w be outside this set of measure zero. Clearly the conclusions of Theorem 2.2 are applicable. To complete the proof of Theorem 2.1, we need only show that none of the paths diverge to infinity.

Hence, suppose for contradiction that a path $(z(\alpha),t(\alpha))$ in $H_w^{-1}(0)$ diverges to infinity as α approaches some $\bar{\alpha}$. Consider, for any i, $G_i\left(\frac{z(\alpha)}{\|z(\alpha)\|},t(\alpha),a,c\right)$. Since G_i is homogeneous in z of degree q_i,

$$
\begin{aligned}
G_i\left(\frac{z(\alpha)}{\|z(\alpha)\|},t(\alpha),a,c\right) &= \|z(\alpha)\|^{-q_i} G_i(z(\alpha),t(\alpha),a,c) \\
&= \|z(\alpha)\|^{-q_i} [G_i(z(\alpha),t(\alpha),a,c) - H_i(z(\alpha),t(\alpha),w)] \\
&= \|z(\alpha)\|^{-q_i} [1 - t(\alpha) + t(\alpha)(Q_i(z(\alpha),a,c) - P_i(z(\alpha),w))] \\
&\to 0 \text{ as } \alpha \to \bar{\alpha}.
\end{aligned}
$$

Hence, with (z,t) a cluster point of $\left(\frac{z(\alpha)}{\|z(\alpha)\|},t(\alpha)\right)$ we have $G(z,t,a,c) = 0$, $\|z\| = 1$, $0 \leq t \leq 1$. Since G is homogeneous, $G(\lambda z,t,a,c) = 0$ for all complex numbers λ. The solution to $G(\cdot,\cdot,a,c) = 0$ is not a path in a neighborhood of (z,t) so that 0 is not a regular value of $G(\cdot,\cdot,a,c)$ on the set $\{(z,t)|z \neq 0\}$, a contradiction. ∎

Observe that the method of proof shows a constructive way of generating all the q solutions for almost any w. Simply use the method described in [7] to follow any of the q paths starting at a trivial solution at $t = 0$. Without fail, this path would lead us to a solution to $P(z,w) = 0$ at $t = 1$.

§3. SUFFICIENCY CONDITIONS.

Suppose we are given a particular polynomial system $P(z,\bar{w}) = 0$. How many solutions will there be to this polynomial system?

The answer, as shown in section 2, is: "q, with probability one" where q is the product of the degrees of the equations. The problem with this answer, however, is that oftentimes, when faced with a particular polynomial system, we need an unconditional answer.

This section shows a sufficient condition for a given polynomial system to have exactly q solutions. Then, we show how the condition can be applied to certain nontrivial polynomial systems.

The theorem we show is

Theorem 3.1. Let $P(z,\bar{w}) = 0$ be given, and let $Q(z,\bar{a},\bar{c}) = 0$ be its corresponding highest ordered system of equations. If $Q(z,\bar{a},\bar{c}) = 0$ has only the trivial solution $z = 0$, then $P(z,\bar{w}) = 0$ has $q \equiv \prod_{i=1}^{n} q_i$ solutions, where q_i is the degree of P_i.

Our method of proof of Theorem 3.1 can be used to prove the classical theorem of Bezout [11] and the theorem of Noether and van der Waerden [9]. The first theorem states that an arbitrary $P(z,\bar{w}) = 0$ has at most q isolated solutions. The second theorem states that under the hypothesis of Theorem 3.1, $P(z,\bar{w}) = 0$ has at most q distinct solutions. (This theorem was rediscovered by S. Friedland [6]). These classical theorems were proved by algebraic approaches.

Let us introduce a definition for multiplicity. Consider an isolated solution z^0 of $P(z,\bar{w}) = 0$. Let N be an open neighborhood of z^0 containing no other solution of $P(z,\bar{w}) = 0$. Let $\deg(P(\cdot,\bar{w}),N,0)$ be the Brouwer degree, where $P(\cdot,\bar{w})$ is regarded on the space R^{2n}, the space induced by C^n in a natural fashion. [2] states that $\deg(P,N,0)$ is always a positive integer. We say z^0 is a solution of multiplicity k if $\deg(P,N,0) = k$. Thus in Theorem 3.1, every solution is counted $k \geq 1$ times.

To prove Theorem 3.1, we need a couple of lemmas.

Lemma 3.2. Let $P(z,\bar{w}) = 0$ be given. If $Q(z,\bar{a},\bar{c}) = 0$ has only the trivial solution, then there is a finite, nonzero number of solutions to $P(z,\bar{w}) = 0$.

Proof: By Theorem 2.1, for almost all $e \in C^m$, $P(z,\bar{w} + e) = 0$ has exactly q solutions. Let $z(e)$ be a solution of $P(z,\bar{w} + e) = 0$. We show that the sequence $\{z(e)\}$ remains in a bounded set as e approaches 0.

For contradiction, suppose $\|z(e)\| \to \infty$ as $e \to 0$. Then

$$
\begin{aligned}
Q_i\left(\frac{z(e)}{\|z(e)\|}, \bar{a},\bar{c}\right) &= \|z(e)\|^{-q_i} Q_i(z(e),\bar{a},\bar{c}) \\
&= \|z(e)\|^{-q_i} [Q_i(z(e),\bar{a},\bar{c}) - P_i(z(e),\bar{w} + e)] \\
&= \|z(e)\|^{-q_i} [Q_i(z(e),\bar{a},\bar{c}) - P_i(z(e),\bar{w}) - P_i(z(e),e)] \\
&\to 0 \text{ as } e \to 0 .
\end{aligned}
$$

Hence, if z is a cluster point of $\frac{z(e)}{\|z(e)\|}$, we have $Q(z,\bar{a},\bar{c}) = 0$, $\|z\| = 1$, a contradiction.

Thus, each $z(e)$ remains bounded as e approaches zero. By continuity, all cluster points of the sequence $\{z(e)\}$ are solutions of $P(z,\bar{w}) = 0$.

Hence, the solution set of $P(z,\bar{w}) = 0$ is nonempty and bounded. But from [10, Corollary 2.2], this implies that each solution is isolated. Therefore, there must be a finite, nonzero number of solutions for $P(z,\bar{w}) = 0$. ∎

Lemma 3.3. Let z^0 be an isolated solution of $P(z,w) = 0$ of multiplicity k. Let N be an open neighborhood of z^0 containing no other solution of $P(z,\bar{w}) = 0$. Then, for all sufficiently small $e \in C^m$ such that 0 is a regular value of $P(\cdot,\bar{w} + e)$, $P(z,\bar{w} + e) = 0$ has k distinct solutions in N.

Proof: Since the Brouwer degree is invariant under small perturbations, for all sufficiently small e, $\deg(P(\cdot,\bar{w} + e),N,0) = k$. Since $D_zP(\cdot,\bar{w} + e)$, regarded on the space R^{2n}, has nonnegative determinant [4], and since 0 is a regular value of $P(\cdot,\bar{w} + e)$, we have that the number of solutions for $P(z,\bar{w} + e) = 0$ in N equals the Brouwer degree, namely k. ∎

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1.

From Lemma 3.2, the solutions for $P(z,\bar{w}) = 0$ are $\{z^1, z^2, \ldots, z^p\}$, for some integer $p \geq 1$. Let z^i have multiplicity k_i, for $i = 1,2,\ldots,p$. By Theorem 2.1, for almost all e, $P(z,\bar{w} + e) = 0$ has q distinct solutions.

But as e approaches zero, these q solutions will tend towards solutions of $P(z,\bar{w}) = 0$. Hence, we have from Lemma 3.3 that $q = \sum_{i=1}^{p} k_i$. ∎

Remarks:

1. If 0 is a regular value of $P(\cdot,\bar{w})$ in Theorem 3.1, then all q solutions will be distinct.

2. Given an arbitrary system $P(z,\bar{w}) = 0$, by Lemma 3.3 and Theorem 2.1, the number of isolated solutions of $P(z,\bar{w}) = 0$ is at most q. This is the classical theorem of Bezout.

3. If $Q(z,\bar{a},\bar{c}) = 0$ has only the trivial solution, then by Lemma 3.2 and Bezout's theorem above, there are at most q distinct solutions of $P(z,\bar{w}) = 0$. This is the classical theorem of Noether and van der Waerden [9].

4. Given an arbitrary $P(z,\bar{w}) = 0$, all isolated solutions can be generated by the method in [7].

5. The hypothesis of Theorem 3.1 can be constructively verified. For there is a classical procedure [11] for determining whether or not a homogeneous system $Q = 0$ has nontrivial solutions.

6. Note that the condition: $Q(z,\bar{a},\bar{c}) = 0$ only has a trivial solution, is a condition on the <u>form</u> in which $P(z,\bar{w}) = 0$ is written. For example, the system

$$2z_1z_2 + z_1 = 1$$
$$z_1z_2 + z_2 = 2$$

has only two solutions, so that $Q = 0$ must have nontrivial solutions, as indeed it does. The equivalent system, however,

$$z_1 - 2z_2 = -3$$
$$z_1z_2 + z_2 = 2$$

has only trivial solutions for $Q = 0$, so that this form of the equations reveals that the system has only two solutions.

Finally, let us show some applications of Theorem 3.1 to certain systems.

Theorem 3.4. Let $P(z,\bar{w}) = 0$ be such that $Q(z,\bar{a},\bar{c}) = 0$ is of the form

$$\sum_{j=1}^{n} e_{ij} z_j^r, \qquad i = 1,2,\ldots,n \tag{3.1}$$

where e_{ij} are complex numbers, and r a positive integer. Then, if $e = (e_{ij})$ is nonsingular, $P(z,\bar{w}) = 0$ has r^n solutions.

Proof: The system $ey = 0$ where $y = (y_1,\ldots,y_n)$ are complex variables has only the trivial solution. Thus, the system $\sum_{j=1}^{n} e_{ij} z_j^r = 0$ all i, has only the trivial solution. ∎

Let $e = (e_{ij}) \in C^{n^2}$. A principal submatrix of e is a submatrix formed by deleting row i and column i, for $i \in I$, where $I \subsetneq \{1,2,\ldots,n\}$ ($I = \phi$ is possible).

Theorem 3.5. Let $P(z,\bar{w}) = 0$ be such that $Q(z,\bar{a},\bar{c}) = 0$ has the form

$$Q_i(z,\bar{a},\bar{c}) = z_i^{s_i} \left(\sum_{j=1}^{n} e_{ij} z_j^r \right). \tag{3.2}$$

If $e = (e_{ij})$ has nonsingular principal submatrices, then $P(z,\bar{w}) = 0$ has $\prod_{i=1}^{n} (r + s_i)$ solutions.

Proof: Consider any z satisfying $Q(z,\bar{a},\bar{c}) = 0$. If $z \neq 0$, then let $z = (z_I, 0)$ where $z_i \neq 0$, all $i \in I$. Since $Q_i(z,\bar{a},\bar{c}) = 0$, all $i \in I$, we must have $\sum_{j \in I} e_{ij} z_j^r = 0$, all $i \in I$, i.e., $e_{II} y_I = 0$ where e_{II} is the principal submatrix formed by deleting row i and column i, for $i \notin I$, and $y_i = z_i^r$, $i \in I$. Since e_{II} is nonsingular, $y_I = 0$, a contradiction.

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Homotopy, path-following, polynomial systems, systems of equations, complex variables, algebraic systems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is shown that for almost every system of n polynomial equations in n complex variables, the number of solutions is equal to $q \equiv \prod_{i=1}^{n} q_i$, where q_i is the degree of equation i. The proof of this result is done

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in such a way that all q solutions can be explicitly calculated for almost all such systems.

It is further shown that if the polynomial system obtained by retaining only the terms of degree q_i in each equation i has only the trivial solution, then the number of solutions is equal to q.